Governing Equations

Fluid Properties

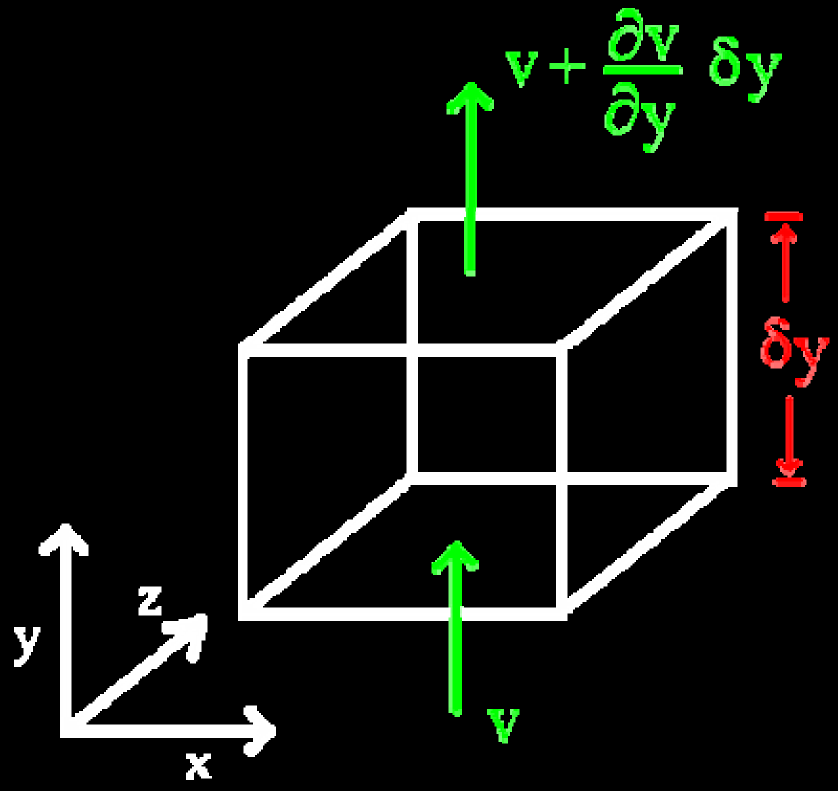
- Properties required to describe flow:
- normal shear stress (pressure)
- viscosity, μ, (gives tangential shear stress)
- density, ρ
- plus, velocity of fluid flow, u,v,w
- temperature of fluid, T

Governing Equations

- Conservation of Mass
- Newton's Second Law of Motion
- First Law of Thermodynamics

Mass

- Consider small volume, δxδyδz in space:



Mass

- Mass of volume = density × volume
- Mass = $\rho\ \delta x \delta y \delta z$
- Assuming volume does not deform, rate of change of mass with time =

$$\frac{\partial \rho}{\partial t} \delta x \delta y \delta z$$

Mass

- Mass into control volume
- Volume flow rate = velocity × area
- Mass flow rate = density × velocity × area
- In x direction = $\rho\ u\ \delta y \delta z$

Mass

- Mass out of system
- Mass flow rate = density × velocity × area
- We need to account for change in velocity and density across the volume
- x velocity out = $u + (\partial u/\partial x)\delta x$
- density out = $\rho + (\partial \rho/\partial x)\delta x$

Mass

- Mass flow out of the control volume

$$\left(\rho u + \frac{\partial \rho u}{\partial x}\delta x\right)\delta y \delta z$$

Mass

- Net mass flow in x direction
- Net flow out = flow out - flow in
- Net flow out = $\dfrac{\partial \rho u}{\partial x}\delta x \delta y \delta z$

Mass

- In three dimensions:
- Net flow out =

$$\left(\frac{\partial \rho u}{\partial x}+\frac{\partial \rho v}{\partial y}+\frac{\partial \rho w}{\partial z}\right)\delta x\delta y\delta z$$

Mass

- In + Made = Out + Accumulated
- Mass cannot be created, so
- In = Out + Accumulated
- Accumulated + Out - In = 0

Mass

- Accumulated + Out - In = 0
- Or, rate of change of mass with time plus net flow of mass out equals zero

$$\frac{\partial \rho}{\partial t}\delta x \delta y \delta z + \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right) \delta x \delta y \delta z = 0$$

Mass

- In the limit of an infinitessimally small volume, $\delta x \delta y \delta z \rightarrow 0$, we can write

$$\frac{\partial \rho}{\partial t} + \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right) = 0$$

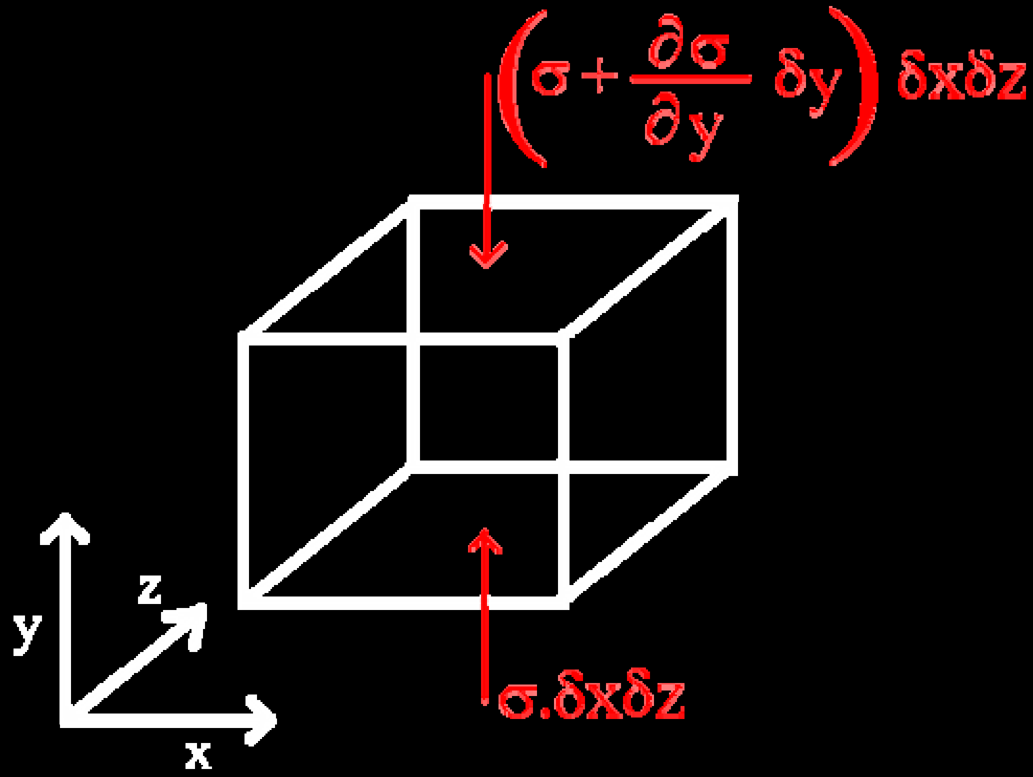
- This is the ***continuity equation***

Momentum

- Use Newton's Second Law to relate forces on a control volume to the acceleration of the fluid
- Forces are shear stresses and normal stresses plus body forces such as gravity

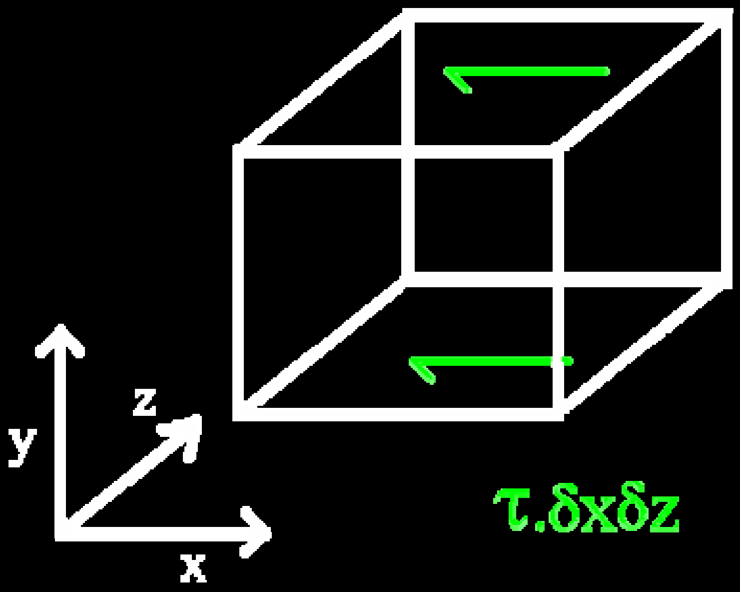
Momentum

- Normal stresses



Momentum

- Shear stresses



$$\left(\tau + \frac{\partial \tau}{\partial y}\,\delta y\right)\delta x \delta z$$

$$\tau.\delta x \delta z$$

Momentum

- Net normal stress in x direction

$$\frac{\partial \sigma_x}{\partial x} \delta x \delta y \delta z$$

- Net shear stress in x direction

$$\left(\frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z} \right) \delta x \delta y \delta z$$

- Plus body force in x direction

Momentum

- Can express acceleration as rate of change of velocity
- Need to consider change of velocity in space and time

Momentum

- Consider velocity in x direction, u

$$\delta u = \left(\frac{\partial u}{\partial x}\delta x + \frac{\partial u}{\partial y}\delta y + \frac{\partial u}{\partial z}\delta z \right) + \frac{\partial u}{\partial t}\delta t$$

- change = change in space + change in time

Momentum

- Divide by δt to give us the change over a small time

$$\frac{\delta u}{\delta t}=\left(\frac{\partial u}{\partial x}\frac{\delta x}{\delta t}+\frac{\partial u}{\partial y}\frac{\delta y}{\delta t}+\frac{\partial u}{\partial z}\frac{\delta z}{\delta t}\right)+\frac{\partial u}{\partial t}$$

- δx/δt is equivalent to u
- δy/δt is equivalent to v
- δz/δt is equivalent to w

Momentum

$$\frac{\delta u}{\delta t} = \left(\frac{\partial u}{\partial x} u + \frac{\partial u}{\partial y} v + \frac{\partial u}{\partial z} w \right) + \frac{\partial u}{\partial t}$$

Acceleration in x direction

Change in velocity due to convection into space

Change in time of x component of velocity

Momentum

- Now, use Newton's Second Law to relate forces on a control volume to the acceleration of the fluid
- Force = Mass × Acceleration
- With Mass = $\rho\, \delta x \delta y \delta z$
- and body force per unit mass = f_x

Momentum

- For each direction (incompressible flow):

$$\rho\left(\frac{\partial u}{\partial x}u+\frac{\partial u}{\partial y}v+\frac{\partial u}{\partial z}w+\frac{\partial u}{\partial t}\right)=\left(\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+\frac{\partial \tau_{xz}}{\partial z}\right)+f_x$$

$$\rho\left(\frac{\partial v}{\partial x}u+\frac{\partial v}{\partial y}v+\frac{\partial v}{\partial z}w+\frac{\partial v}{\partial t}\right)=\left(\frac{\partial \sigma_y}{\partial y}+\frac{\partial \tau_{yx}}{\partial x}+\frac{\partial \tau_{yz}}{\partial z}\right)+f_y$$

$$\rho\left(\frac{\partial w}{\partial x}u+\frac{\partial w}{\partial y}v+\frac{\partial w}{\partial z}w+\frac{\partial w}{\partial t}\right)=\left(\frac{\partial \sigma_z}{\partial z}+\frac{\partial \tau_{zx}}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}\right)+f_z$$

Momentum

- Normal stresses and shear stresses relate to the fluid pressure and viscosity:
- For Newtonian fluids:

$$\sigma_x = -p + 2\mu \frac{\partial u}{\partial x}$$

$$\tau_{xy} = \mu \left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \right)$$

Momentum

- The Navier-Stokes equation in x direction:

$$\rho\left(\frac{\partial u}{\partial x}u+\frac{\partial u}{\partial y}v+\frac{\partial u}{\partial z}w+\frac{\partial u}{\partial t}\right)=\frac{\partial}{\partial x}\left(-p+2\mu\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)\right]+\frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right)\right]+f_x$$

- Equivalent equations for y and z directions

Momentum

- The body force is the force due to gravity on the control volume
- If gravity acts in the negative y direction (this is the default in FLOTHERM)

$$f_y = \frac{1}{\rho} g(\rho_{ref} - \rho)\delta x \delta y \delta z$$

Momentum

- We can use the Boussinesq approximation to write the body force as:

$$f_y = g\beta\left(T - T_{ref}\right)\delta x \delta y \delta z$$

- With

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p$$

Energy

- First Law of Thermodynamics
- Energy is Conserved

- Rate of change of energy within the element = Net flux of heat into the element + rate of work done on element due to body and surface forces + Source terms

Energy

- Energy of a fluid, E
- Internal (thermal) energy, I
- Kinetic energy, $\frac{1}{2}\left(u^2 + v^2 + w^2\right)$
- Potential energy (due to gravity)

Energy

- Net energy change in fluid = sum of work done on fluid + net rate of heat addition + energy sources
- Include potential energy as a source term

Energy

- Work done on fluid:
- Rate of work done on fluid element by surface force equals product of force and component of velocity in direction of force
- In the x direction:

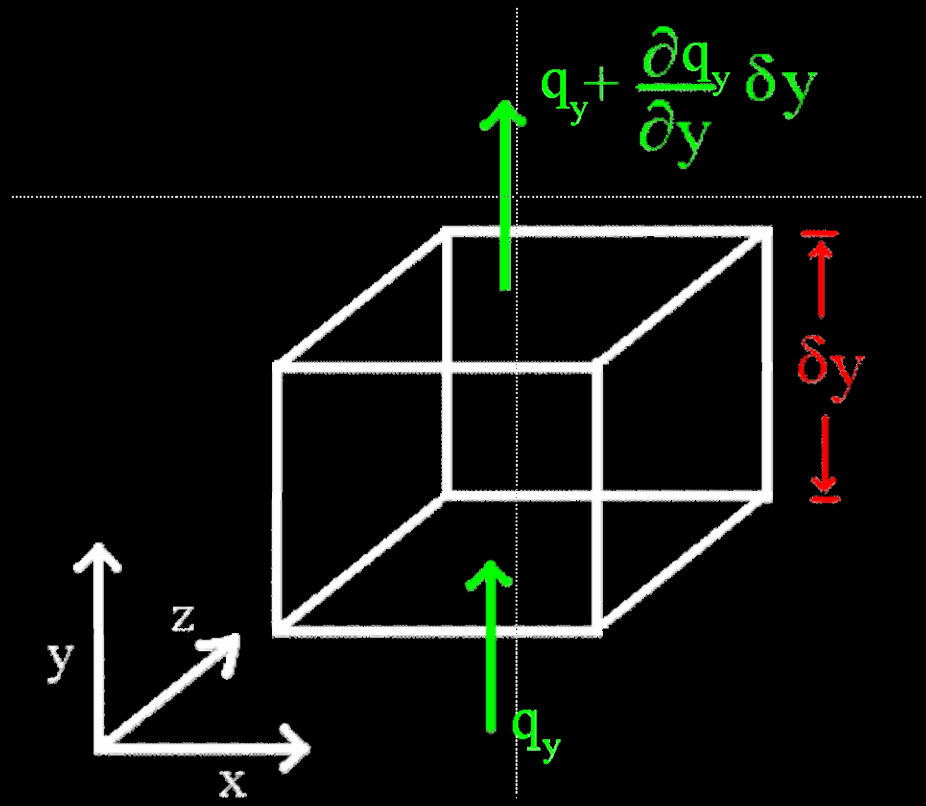
$$\left(\frac{\partial [u(-p + \tau_{xx})]}{\partial x} + \frac{\partial (u\tau_{xy})}{\partial y} + \frac{\partial (u\tau_{xz})}{\partial z} \right) \delta x \delta y \delta z$$

Energy

- Can write equations for each direction and find the total rate of work done on the fluid particle by surface stresses

Energy

- Net rate of heat transfer to the fluid particle:



Energy

- Heat into fluid particle:

$$q_x \delta y \delta z$$

- Heat out of fluid particle:

$$\left(q_x + \frac{\partial q_x}{\partial x} \delta x \right) \delta y \delta z$$

Energy

- Net rate of heat addition:

$$-\frac{\partial q_x}{\partial x}\delta x\delta y\delta z$$

- Total rate of heat addition per unit volume:

$$-\frac{\partial q_x}{\partial x}-\frac{\partial q_y}{\partial y}-\frac{\partial q_z}{\partial z}$$

Energy

- Fourier's Law of Conduction gives:

$$q_x = -k \frac{\partial T}{\partial x}$$

Energy

- In vector notation, the energy equation:

$$\rho\frac{DE}{DT} = -\mathrm{div}(p\vec{u}) + \left(\frac{\partial(u\tau_{xx})}{\partial x} + \frac{\partial(u\tau_{xy})}{\partial y} + \frac{\partial(u\tau_{xz})}{\partial z} + \frac{\partial(v\tau_{xy})}{\partial x} + \frac{\partial(v\tau_{yy})}{\partial y}\right) +$$

$$\left(\frac{\partial(v\tau_{yz})}{\partial z} + \frac{\partial(w\tau_{xz})}{\partial x} + \frac{\partial(w\tau_{yz})}{\partial y} + \frac{\partial(w\tau_{zz})}{\partial z}\right) + div(k\,grad\,T) + S_E$$

Governing Equations

- We now have enough equations to fully describe our system
- Mass
- Momentum
- Energy

General Transport Equation

- We can write a general transport equation, for a general variable, ϕ.
- Use vector notation:

$$div(\rho U) = \left(\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} + \frac{\partial \rho w}{\partial z} \right)$$

$$grad(\varphi) = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right)$$

General Transport Equation

- General equation:

$$\frac{\partial}{\partial t}(\rho\varphi) + div(\rho U\varphi - \Gamma grad\varphi) = S_\varphi$$

transient + convection - diffusion = source